数理解析研究所講究録
1156 巻 2000 年 35-52

ディラック作用素の境界値逆問題について

九州大学大学院数理学研究科
土田 哲生 (Tetsuo Tsuchida)
(中村 玄 教授 (群馬大学工学部) との共同研究)

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^3$ with connected smooth boundary $\partial\Omega$. We consider a Dirac operator

$$L_{\vec{a},q}u = \begin{pmatrix} q_+I_2 & \sigma\cdot(D+\vec{a}) \\ \sigma\cdot(D+\vec{a}) & q_-I_2 \end{pmatrix}\begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$
$$= \begin{pmatrix} q_+(x)I_2 & \sum_{j=1}^3 \sigma_j(D_j+a_j(x)) \\ \sum_{j=1}^3 \sigma_j(D_j+a_j(x)) & q_-(x)I_2 \end{pmatrix}\begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \tag{1.1}$$

where $x=(x_1,x_2,x_3)\in\Omega$, $D=(D_1,D_2,D_3)$ with $D_j=-i\partial/\partial x_j$, $j=1,2,3$, and $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ is the vector of Pauli matrices, i.e.,

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

Let the scalar potential $q(x)=(q_+(x),q_-(x))$ and the vector potential $\vec{a}(x)=(a_1(x), a_2(x),a_3(x))$ be $\mathbf{R}^2$- and $\mathbf{R}^3$-valued $C^\infty(\bar{\Omega})$ functions, respectively. We define a selfadjoint operator $L^{(+)}_{\vec{a},q}$ on $(L^2(\Omega))^4$ by $L^{(+)}_{\vec{a},q}u=L_{\vec{a},q}u$ for $u\in D(L^{(+)}_{\vec{a},q})$ with domain

$$D(L^{(+)}_{\vec{a},q})=\{\begin{pmatrix} u_+ \\ u_- \end{pmatrix}\in(L^2(\Omega))^2\times(L^2(\Omega))^2 | u_+\in(H^1_0(\Omega))^2,\ u_-\in\mathcal{H}(\Omega)\},$$

where $\mathcal{H}(\Omega)=\overline{(H^1(\Omega))^2}^{\|\sigma\cdot D\cdot\|+\|\cdot\|}$, with $\|\cdot\|=\|\cdot\|_{(L^2(\Omega))^2}$.

Consider a Dirichlet boundary value problem

$$\begin{cases} L_{\vec{a},q}u=\begin{pmatrix} q_+I_2 & \sigma\cdot(D+\vec{a}) \\ \sigma\cdot(D+\vec{a}) & q_-I_2 \end{pmatrix}\begin{pmatrix} u_+ \\ u_- \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{in } \Omega, \\ u_+|_{\partial\Omega}=f\in h(\partial\Omega), \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

here $h(\partial\Omega)$ is the trace space on $\partial\Omega$ of $\mathcal{H}(\Omega)$. If $0\in\rho(L^{(+)}_{\vec{a},q})$ (resolvent set of $L^{(+)}_{\vec{a},q}$), then for any boundary value $f\in h(\partial\Omega)$, there exists a unique solution $u=(u_+,u_-)\in\mathcal{H}(\Omega)\times\mathcal{H}(\Omega)$ to (1.3). Define a Dirichlet to Dirichlet map $\Lambda_{\vec{a},q}$ on $h(\partial\Omega)$, by

$$\Lambda_{\vec{a},q}f=u_-|_{\partial\Omega}\in h(\partial\Omega), \qquad \text{for } f\in h(\partial\Omega),$$

where $u=(u_+,u_-)$ is the unique solution of (1.3). See [NT] for details. Note that the D-D map $\Lambda_{\vec{a},q}$ is invariant under a gauge transformation in the vector potential: if $p\in W^{1,\infty}(\Omega)$ and $p|_\Gamma=0$, then $\Lambda_{\vec{a}+\nabla p,q}=\Lambda_{\vec{a},q}$.

The principal aim of this paper is to show that $\Lambda_{\vec{a},q}$ determines $\mathrm{rot}\vec{a}$ and q uniquely. In the following statements we always assume $\vec{a}_j, q_j = (q_{j,+}, q_{j,-}) \in C^\infty(\overline{\Omega})$, $j = 1, 2$.

Theorem 1. *Assume* $\vec{a}_1 = \vec{a}_2$ *to infinite order at* Γ *and* $0 \in \rho(L^{(+)}_{\vec{a}_j, q_j})$, $j = 1, 2$. *If* $\Lambda_{\vec{a}_1,q_1} = \Lambda_{\vec{a}_2,q_2}$, *then* $\mathrm{rot}\vec{a}_1 = \mathrm{rot}\vec{a}_2$ *and* $q_1 = q_2$ *in* Ω.

Theorem 2. *Assume* $0 \in \rho(L^{(+)}_{\vec{a}_j, q_j})$, $j = 1, 2$. *If* $\Lambda_{\vec{a}_1,q_1} = \Lambda_{\vec{a}_2,q_2}$, *then we can find* $p \in C^\infty(\overline{\Omega})$ *vanishing to first order at* $\partial\Omega$ *such that* $\vec{a}_1 = \vec{a}_2 + \nabla p$ *to infinite order at* $\partial\Omega$. As a corollary of Theorem 1 and 2, we have

Corollary 3. *If* $\Lambda_{\vec{a}_1,q_1} = \Lambda_{\vec{a}_2,q_2}$, *then* $\mathrm{rot}\vec{a}_1 = \mathrm{rot}\vec{a}_2$ *and* $q_1 = q_2$ *in* Ω.

Next let us give a theorem about an inverse scattering problem. Rewrite (1.1) in the form:

$$L_V u = L_{\vec{a},q} u = \left[\begin{pmatrix} I_2 & \sigma \cdot D \\ \sigma \cdot D & -I_2 \end{pmatrix} + V(x) \right] \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \tag{1.4}$$

where we have extended $\vec{a}, q$ to the whole $\mathbf{R}^3$ such that $\vec{a}$ and q in (1.1) are absorbed into a compactly supported Hermitian matrix V whose components are in $C_0^\infty(\mathbf{R}^3)$.

Define an orthonormal system $(b_1^+(\xi), b_2^+(\xi), b_1^-(\xi), b_2^-(\xi))$ in $\mathbf{C}^4$ by

$$(b_1^+(\xi), b_2^+(\xi), b_1^-(\xi), b_2^-(\xi)) := \begin{pmatrix} a_+(\xi) I_2 & -a_-(\xi)\sigma \cdot \frac{\xi}{|\xi|} \\ a_-(\xi)\sigma \cdot \frac{\xi}{|\xi|} & a_+(\xi) I_2 \end{pmatrix} \tag{1.5}$$

with $a_\pm(\xi) := \sqrt{\frac{1}{2}(1 \pm \frac{1}{<\xi>})}$, $<\xi> := \sqrt{1 + |\xi|^2}$, $\xi \in \mathbf{R}^3$. For θ in the unit sphere S^2 centered at the origin and $\pm E > 1$, consider the unique solution $\psi = \psi(x, \theta; E)$ to

$$(L_V - E)\psi = 0 \qquad \text{in} \qquad \mathbf{R}^3 \tag{1.6}$$

such that each component v of

$$\psi^s := \psi - e^{i\nu(E)\theta\cdot x}(b_1^\pm(\nu(E)\theta), b_2^\pm(\nu(E)\theta)) \quad (\pm E > 1)$$

is outgoing (i.e. (*) $(\partial/\partial r \mp i\nu(E))v = o(r^{-1})$ $(r = |x| \to \infty)$ $\pm E > 1$ with $\nu(E) := \sqrt{E^2 - 1}$ and (**) $v = O(r^{-1})$ $(r \to \infty)$).

Note that

$$(L_V - V - E)(\psi - \psi^s) = 0. \tag{1.7}$$

Then, by (1.7) and the integral representation of ψ^s, $\psi^s = \psi^s(x, \theta; E)$ has the asymptotic property:

$$\psi^s(x, \theta; E) = -\frac{e^{\pm i\nu(E)r}}{4\pi r}\psi^\infty(\frac{x}{|x|}, \theta; E) + o(r^{-1}) \quad (r \to \infty) \quad \text{for} \quad \pm E > 1. \tag{1.8}$$

Define the scattering amplitude $A_V(E) : (L^2(S^2))^2 \to (L^2(S^2))^2$, as the operator with the integral kernel:

$$a_V(\theta', \theta; E) := (b_1^\pm(\nu(E)\theta'), b_2^\pm(\nu(E)\theta'))^* \psi^\infty(\theta', \theta; E), \quad \theta, \theta' \in S^2; \pm E > 1.$$

Then, we have the following uniqueness result for the inverse scattering problem at fixed energy E.

Theorem 4. *Let $\Omega \subset \mathbf{R}^3$ be a bounded smooth domain with connected exterior $\Omega^e = \mathbf{R}^3 \setminus \bar{\Omega}$. Let V_j $(j=1,2)$ be Hermitian matrices associated with $\vec{a}_j, q_j$ $(j=1,2)$ whose components are in $C_0^\infty(\mathbf{R}^3)$ and assume that $V_1 = V_2$ in $\mathbf{R}^3 \setminus \Omega$ and $E \in \rho(L_{V_j}^{(+)})$ $(j=1,2)$. Then $A_{V_1}(E) = A_{V_2}(E)$ is equivarent to $\Lambda_{V_1-E} = \Lambda_{V_2-E}$. Hence $A_{V_1}(E) = A_{V_2}(E)$ implies* $\mathrm{rot}\vec{a}_1 = \mathrm{rot}\vec{a}_2$, $q_1 = q_2$ *in Ω.*

For Schrödinger operators with magnetic potential $\vec{a}$ and electrical potential q on $\Omega \subset \mathbf{R}^n$, $n \geq 3$, the Dirichlet-Neumann map determines $\mathrm{rot}\vec{a}$ and q uniquely ([Su], [NSU]). For Dirac operators, the cases where potentials are small were treated in [T1]. The reconstruction of the scalar potential and magnetic fields of Dirac operator from the scattering amplitude is given in [I],[G].

Here we will sketch the proofs of Theorem 1 and 2. For the details, see [NT] and [T2].

2. Proof of Theorem 1

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and α_4 be 4×4 Hermitian matrices:

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1,2,3, \quad \alpha_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Then we can see the anti-commutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \quad j,k = 1,2,3,4. \tag{2.1}$$

Let

$$P_\pm = (I_4 \pm \alpha_4)/2 \tag{2.2}$$

be orthogonal projections on $\mathbf{C}^4$ and write

$$q(x) := \begin{pmatrix} q_+(x) I_2 & 0 \\ 0 & q_-(x) I_2 \end{pmatrix} = q_+(x) P_+ + q_-(x) P_-,$$

and then Dirac operator can be written as

$$L_{\vec{a},q} = \alpha \cdot (D + \vec{a}) + q.$$

In this paper we use the following relations by (2.1): for any $a, b \in \mathbf{C}^3$,

$$\alpha \cdot a \alpha \cdot b + \alpha \cdot b \alpha \cdot a = 2a \cdot b I_4, \text{ in particular } (\alpha \cdot a)^2 = a^2 I_4, \tag{2.3}$$

$$\alpha \cdot a P_\pm = P_\mp \alpha \cdot a, \tag{2.4}$$

$$\alpha \cdot a q = q^I \alpha \cdot a \quad \text{with } q^I := q_+(x) P_- + q_-(x) P_+. \tag{2.5}$$

We omit "$\rightarrow$" of the vector potential $\vec{a}(x)$ in §2.

Lemma 2.1. *For any solution* $u^{(j)} = (u_+^{(j)}, u_-^{(j)}) \in \mathcal{H}(\Omega) \times \mathcal{H}(\Omega)$ *of* $L_{a_j,q_j} u^{(j)} = 0$, $j = 1, 2$, *it follows that*

$$_{h(\Gamma)} < \overline{u_+^{(2)}}, i\sigma \cdot N(\Lambda_{a_1,q_1} - \Lambda_{a_2,q_2}) u_+^{(1)} >_{h(\Gamma)^*} = \int_\Omega {}^t\overline{u^{(2)}} \cdot (V_1 - V_2) u^{(1)}\, dx,$$

where $V_j = \alpha \cdot a_j + q_j$, $j = 1, 2$, *and* N *is the unit outer normal vector on* Γ. *In particular if* $\Lambda_{a_1,q_1} = \Lambda_{a_2,q_2}$, *then*

$$\int_\Omega {}^t\overline{u^{(2)}} \cdot (V_1 - V_2) u^{(1)}\, dx = 0. \tag{2.6}$$

Proof is omitted.

In what follows we assume $a, q \in C_0^\infty(\mathbf{R}^3)$. ($a, q$ are regarded as extensions of $a_j, q_j \in C^\infty(\overline{\Omega})$). Let $Z = \{\zeta \in \mathbf{C}^3 | \, \zeta^2 = \zeta \cdot \zeta = 0, \, |\zeta| \geq 1\}$. We look for a solution of $L_{a,q} u = 0$ of the form: with 4×4-matrix-valued functions u_ζ, v_ζ,

$$u_\zeta(x) = e^{i\zeta \cdot x} v_\zeta(x), \quad x \in \mathbf{R}^3, \, \zeta \in Z. \tag{2.7}$$

Hence v_ζ satisfies

$$(\alpha \cdot (D + \zeta) + \alpha \cdot a + q) v_\zeta = 0. \tag{2.8}$$

Step 1. Intertwining property.

We consider operators M_ζ and Δ_ζ:

$$M_\zeta := (\alpha \cdot (D + \zeta) + \alpha \cdot a + q)(\alpha \cdot (D + \zeta) + \alpha \cdot a - q^I), \tag{2.9}$$

$$\Delta_\zeta := (D + \zeta)^2 = -\Delta + 2\zeta \cdot D. \tag{2.10}$$

Then using (2.3,5), we have

$$\begin{aligned} M_\zeta &= (D + \zeta)^2 I_4 + 2a \cdot (D + \zeta) I_4 \\ &\qquad + [\alpha \cdot D(\alpha \cdot a - q^I) + (\alpha \cdot a + q)(\alpha \cdot a - q^I)] \\ &= \Delta_\zeta I_4 + 2a \cdot (D + \zeta) I_4 + W, \\ &\text{where} \quad W = \alpha \cdot D(\alpha \cdot a - q^I) + (\alpha \cdot a + q)(\alpha \cdot a - q^I). \end{aligned} \tag{2.11}$$

We use pseudodifferential operators depending on a parameter $\zeta \in Z$. We denote by $S^m(Z) = S^m(\mathbf{R}^3, Z)$ the space of symbols of order m in the Shubin class and by $L^m(Z) = L^m(\mathbf{R}^3, Z)$ the space of Ps.D.O. of order m (see [NU]). If $a_\zeta(x, \xi) \in S^m(Z)$ is positive homogeneous of degree m in (ζ, ξ), i.e. $a_{t\zeta}(x, t\xi) = t^m a_\zeta(x, \xi)$ for $t > 0$, $\zeta, t\zeta \in Z$, $\xi \in \mathbf{R}^3$, we write $a_\zeta(x, \xi) \in HS^m(Z)$.

Put $\lambda_\zeta(\xi) := (|\xi|^2 + |\zeta|^2)^{1/2}$ and let $\Lambda_\zeta^s \in L^s(Z), s \in \mathbf{R}$ be a properly supported Ps.D.O. with principal symbol $\sigma(\Lambda_\zeta^s) = \lambda_\zeta^s(\xi)$. For the definition of properly supported, see [NU]. Put $\tilde{M}_\zeta := M_\zeta \Lambda_\zeta^{-1}$ and $\tilde{\Delta}_\zeta := \Delta_\zeta \Lambda_\zeta^{-1}$.

Lemma 2.2. *For any positive integer* N, *there exist elliptic properly supported* $A_\zeta, B_\zeta \in L^0(Z)$ *such that*

$$\tilde{M}_\zeta A_\zeta = B_\zeta \tilde{\Delta}_\zeta + R_\zeta^{(-N)}, \quad R_\zeta^{(-N)} \in L^{-N}(Z). \tag{2.12}$$

Proof. This lemma is essentially the same as Theorem 1.23 in [NU] or Lemma 3.16 in [NSU]. Let $q_\zeta(\xi)$ be the principal symbol of $\tilde{\Delta}_\zeta$:

$$q_\zeta(\xi) := \sigma(\tilde{\Delta}_\zeta) = (\xi+\zeta)^2\lambda_\zeta^{-1}(\xi)$$

and put $\mathcal{M} = \{\xi \in \mathbf{R}^3 | \, q_\zeta(\xi) = 0\}$. Then

$$\mathcal{M} = \{\xi \in \mathbf{R}^3 | \, \mathrm{Im}\zeta \cdot \xi = 0, |\xi + \mathrm{Re}\zeta| = |\mathrm{Re}\zeta|\}$$

and there exists $\varepsilon > 0$ such that

$$\mathrm{Re}\partial_\xi q_\zeta(\xi) \text{ and } \mathrm{Im}\partial_\xi q_\zeta(\xi) \text{ are linearly independent on } N_{5\varepsilon|\zeta|}(\mathcal{M}),$$

where $N_R(\mathcal{M})$ is an R-tubular neighborhood of $\mathcal{M}$.

Set $U_{\zeta,2} = N_{3\varepsilon|\zeta|}(\mathcal{M})$ and $U_{\zeta,1} = \mathbf{R}^3 \setminus N_{2\varepsilon|\zeta|}(\mathcal{M})$. We construct A_ζ, B_ζ as

$$\tilde{\sigma}(A_\zeta)(x,\xi) = \sum_{j=1}^{2} A_{\zeta,j}(x,\xi)\chi_{\zeta,j}(\xi), \quad \tilde{\sigma}(B_\zeta)(x,\xi) = \sum_{j=1}^{2} B_{\zeta,j}(x,\xi)\chi_{\zeta,j}(\xi)$$

with $A_{\zeta,j}, B_{\zeta,j} \in S^0(Z)$. Here $\chi_{\zeta,j}(\xi) \in HS^0(Z)$ is a partition of unity subordinate to $U_{\zeta,j}, j = 1,2$.

First we construct $A_{\zeta,2}$ and $B_{\zeta,2}$ as $A_{\zeta,2} = B_{\zeta,2}$. Take $\psi_{\zeta,1}(\xi) \in C_0^\infty(N_{5\varepsilon|\zeta|}(\mathcal{M})) \cap HS^0(Z)$ such as $\psi_{\zeta,1} = 1$ on $N_{4\varepsilon|\zeta|}(\mathcal{M})$ and $\psi_{\zeta,2}(\xi) \in C_0^\infty(N_{4\varepsilon|\zeta|}(\mathcal{M})) \cap HS^0(Z)$ such as $\psi_{\zeta,2} = 1$ on $N_{3\varepsilon|\zeta|}(\mathcal{M})$. Let $N_\zeta^{(0)}(x,\xi)$ be the principal symbol of $\Lambda_\zeta^{-1}2a\cdot(D+\zeta)$:

$$N_\zeta^{(0)}(x,\xi) := \sigma(\Lambda_\zeta^{-1}2a\cdot(D+\zeta)) = 2\lambda_\zeta^{-1}(\xi)a(x)\cdot(\xi+\zeta) \in HS^0(Z).$$

From the composition formula of Ps.D.O. we seek symbols $A_\zeta^{(-k)}(x,\xi), k = 0,1,\dots,N-1$, satisfying the following differential equations:

$$\begin{cases} H_{q_\zeta}A_\zeta^{(0)}(x,\xi) + \psi_{\zeta,1}(\xi)N_\zeta^{(0)}(x,\xi)A_\zeta^{(0)}(x,\xi) = 0 \\ A_\zeta^{(0)}(x,\xi) = I_4, \quad \text{if } \xi \notin \mathrm{supp}\psi_{\zeta,1}, \\ A_\zeta^{(0)}(x,\xi) \to I_4, \quad \text{as } |x| \to \infty, \end{cases} \tag{2.13}$$

and for $k = 1,2,\dots,N-1$,

$$\begin{cases} H_{q_\zeta}A_\zeta^{(-k)}(x,\xi) + \psi_{\zeta,1}(\xi)N_\zeta^{(0)}(x,\xi)A_\zeta^{(-k)}(x,\xi) \\ \qquad\qquad +\psi_{\zeta,1}(\xi)\sigma(J_\zeta^{(-k)})(x,\xi) = 0, \\ \text{where} \\ J_\zeta^{(-k)} = (J_\zeta^{(-k+1)} + \tilde{M}_\zeta A_\zeta^{(-k+1)}(x,D) - A_\zeta^{(-k+1)}(x,D)\tilde{\Delta}_\zeta)\psi_{\zeta,2}(D), \\ J_\zeta^{(0)} = 0, \\ A_\zeta^{(-k)}(x,\xi) = 0, \quad \text{if } \xi \notin \mathrm{supp}\psi_{\zeta,1}, \\ A_\zeta^{(-k)}(x,\xi) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{2.14}$$

Here $H_{q_\zeta} = \partial_\xi q_\zeta \cdot D_x$. We can take a solution of (2.13) such as

$$A_\zeta^{(0)}(x,\xi) = e^{-c_\zeta(x,\xi)} I_4, \tag{2.15}$$

$$\text{with} \quad c_\zeta(x,\xi) = \mathcal{F}^{-1}_{\xi'\to x}\Big[\frac{\psi_{\zeta,1}(\xi)\mathcal{F}_{x\to\xi'}(N_\zeta^{(0)}(x,\xi))}{\partial_\xi q_\zeta \cdot \xi'}\Big]$$
$$= \frac{2}{\pi}\int_{\mathbf{R}^2} (y_1 + iy_2)^{-1}\psi_{\zeta,1}(\xi) N_\zeta^{(0)}(x - y_1 a - y_2 b, \xi)\, dy_1 dy_2, \tag{2.16}$$

here the last equality holds since $a := \mathrm{Re}\partial_\xi q_\zeta$ and $b := \mathrm{Im}\partial_\xi q_\zeta$ are linearly independent on $\mathrm{supp}\psi_{\zeta,1}$. So we can see $A_\zeta^{(0)}(x,\xi) \in HS^0(Z)$. It follows that $J_\zeta^{(-1)} \in L^{-1}(Z)$, since, for the full symbol of $J_\zeta^{(-1)}$,

$$\begin{aligned}
&\tilde\sigma(J_\zeta^{(-1)})(x,\xi) = \tilde\sigma(\tilde M_\zeta A_\zeta^{(0)}(x,D) - A_\zeta^{(0)}(x,D)\tilde\Delta_\zeta)\psi_{\zeta,2}(\xi),\\
&\equiv (q_\zeta(\xi)A_\zeta^{(0)}(x,\xi) - A_\zeta^{(0)}(x,\xi)q_\zeta(\xi))\psi_{\zeta,2}(\xi)\\
&\quad + (\partial_\xi q_\zeta(\xi)\cdot D_x A_\zeta^{(0)}(x,\xi) + N_\zeta^{(0)}(x,\xi)A_\zeta^{(0)}(x,\xi))\psi_{\zeta,2}(\xi) \mod S^{-1}(Z)\\
&= 0.
\end{aligned}$$

We take a solution of (2.14) such as, for $k = 1, 2, \ldots, N-1$,

$$A_\zeta^{(-k)}(x,\xi) = -e^{-c_\zeta(x,\xi)}\mathcal{F}^{-1}_{\xi'\to x}\Big[\frac{\psi_{\zeta,1}(\xi)\mathcal{F}_{x\to\xi'}(e^{c_\zeta(x,\xi)}\sigma(J_\zeta^{(-k)})(x,\xi))}{\partial_\xi q_\zeta\cdot\xi'}\Big].$$

We can see that $A_\zeta^{(-k)}(x,\xi) \in HS^{-k}(Z)$, $1 \le k \le N-1$, and $J^{(-k)} \in L^{-k}(Z)$, $1 \le k \le N$, inductively. Moreover the following holds

$$\begin{aligned}
J^{(-N)} &= (J^{(-N+1)} + \tilde M_\zeta A_\zeta^{(-N+1)}(x,D) - A_\zeta^{(-N+1)}(x,D)\tilde\Delta_\zeta)\psi_{\zeta,2}(D)\\
&= (J^{(-N+2)} + \tilde M_\zeta A_\zeta^{(-N+2)}(x,D) - A_\zeta^{(-N+2)}(x,D)\tilde\Delta_\zeta)\psi_{\zeta,2}^2(D)\\
&\qquad + (\tilde M_\zeta A_\zeta^{(-N+1)}(x,D) - A_\zeta^{(-N+1)}(x,D)\tilde\Delta_\zeta)\psi_{\zeta,2}(D)\\
&\ \ \vdots\\
&= (J^{(-1)} + \tilde M_\zeta A_\zeta^{(-1)}(x,D) - A_\zeta^{(-1)}(x,D)\tilde\Delta_\zeta)\psi_{\zeta,2}^{N-1}(D)\\
&\qquad + (\tilde M_\zeta A_\zeta^{(-2)}(x,D) - A_\zeta^{(-2)}(x,D)\tilde\Delta_\zeta)\psi_{\zeta,2}^{N-2}(D)\\
&\qquad\ \ \vdots\\
&\qquad + (\tilde M_\zeta A_\zeta^{(-N+1)}(x,D) - A_\zeta^{(-N+1)}(x,D)\tilde\Delta_\zeta)\psi_{\zeta,2}(D)\\
&= \tilde M_\zeta \sum_{k=0}^{N-1} A_\zeta^{(-k)}(x,D)\psi_{\zeta,2}^{N-k}(D) - \sum_{k=0}^{N-1} A_\zeta^{(-k)}(x,D)\tilde\Delta_\zeta\psi_{\zeta,2}^{N-k}(D)\\
&= \tilde M_\zeta\Big(\sum_{k=0}^{N-1} A_\zeta^{(-k)}(x,D)\psi_{\zeta,2}^{N-k}(D)\Big) - \Big(\sum_{k=0}^{N-1} A_\zeta^{(-k)}(x,D)\psi_{\zeta,2}^{N-k}(D)\Big)\tilde\Delta_\zeta,
\end{aligned}$$

where we have used $\tilde{\Delta}_\zeta \psi_{\zeta,2}(D) = \psi_{\zeta,2}(D)\tilde{\Delta}_\zeta$ in the last equality. Hence putting

$$A_{\zeta,2}(x,\xi) = B_{\zeta,2}(x,\xi) = \sum_{k=0}^{N-1} A_\zeta^{(-k)}(x,\xi)\psi_{\zeta,2}^{N-k}(\xi), \tag{2.17}$$

we have

$$\tilde{M}_\zeta A_{\zeta,2}(x,D)\chi_{\zeta,2}(D) - B_{\zeta,2}(x,D)\chi_{\zeta,2}(D)\tilde{\Delta}_\zeta = J^{(-N)}\chi_{\zeta,2}(D) \in L^{-N}(Z). \tag{2.18}$$

Next we construct $A_{\zeta,1}(x,\xi)$ and $B_{\zeta,1}(x,\xi)$. Take $\psi_{\zeta,3}(\xi) \in C^\infty(\mathbf{R}^3) \cap HS^0(Z)$ such as $\psi_{\zeta,3} = 0$ on $N_{\varepsilon|\zeta|}(\mathcal{M})$ and $\psi_{\zeta,3} = 1$ on $\mathbf{R}^3 \setminus N_{2\varepsilon|\zeta|}(\mathcal{M})$. We define $B_\zeta^{(-k)}(x,\xi)$, $k = 0,1,\dots,N$, by

$$\begin{cases} B_\zeta^{(0)}(x,\xi) = A_\zeta^{(0)}(x,\xi), \\ B_\zeta^{(-k)}(x,\xi) = \psi_{\zeta,3}(\xi)q_\zeta^{-1}(\xi)\sigma(I_\zeta^{(-k+1)})(x,\xi), \quad k = 1,\dots,N, \\ \text{where} \\ I_\zeta^{(0)} = \tilde{M}_\zeta A_\zeta^{(0)}(x,D) - A_\zeta^{(0)}(x,D)\tilde{\Delta}_\zeta, \\ I_\zeta^{(-k)} = I_\zeta^{(-k+1)}\psi_{\zeta,3}(D) - B_\zeta^{(-k)}(x,D)\tilde{\Delta}_\zeta, \quad k = 1,\dots,N. \end{cases}$$

It is clear that $I_\zeta^{(0)} \in L^0(Z)$ and $B_\zeta^{(-1)}(x,\xi) \in HS^{-1}(Z)$, since $\psi_{\zeta,3}(\xi)\,q_\zeta^{-1}(\xi) \in HS^{-1}(Z)$. Note that

$$\begin{aligned} \tilde{\sigma}(I_\zeta^{(-1)}) &= \tilde{\sigma}(I_\zeta^{(0)}\psi_{\zeta,3}(D)) - \tilde{\sigma}(B_\zeta^{(-1)}(x,D)\tilde{\Delta}_\zeta) \\ &\equiv \sigma(I_\zeta^{(0)})(x,\xi)\psi_{\zeta,3}(\xi) - B_\zeta^{(-1)}(x,\xi)q_\zeta(\xi) \mod S^{-1}(Z) \\ &= 0, \end{aligned}$$

so $I_\zeta^{(-1)} \in L^{-1}(Z)$ and hence $B_\zeta^{(-2)}(x,\xi) \in HS^{-2}(Z)$. In this way, we get $I_\zeta^{(-k)} \in L^{-k}(Z)$ and $B_\zeta^{(-k)}(x,\xi) \in HS^{-k}(Z)$, $k = 1,\dots,N$, inductively. Moreover the following holds

$$\begin{aligned} I_\zeta^{(-N)} &= I_\zeta^{(-N+1)}\psi_{\zeta,3}(D) - B_\zeta^{(-N)}(x,D)\tilde{\Delta}_\zeta \\ &= (I_\zeta^{(-N+2)}\psi_{\zeta,3}(D) - B_\zeta^{(-N+1)}(x,D)\tilde{\Delta}_\zeta)\psi_{\zeta,3}(D) - B_\zeta^{(-N)}(x,D)\tilde{\Delta}_\zeta \\ &\;\vdots \\ &= I_\zeta^{(0)}\psi_{\zeta,3}^N(D) - B_\zeta^{(-1)}(x,D)\tilde{\Delta}_\zeta\psi_{\zeta,3}^{N-1}(D) - B_\zeta^{(-2)}(x,D)\tilde{\Delta}_\zeta\psi_{\zeta,3}^{N-2}(D) \\ &\qquad - \cdots - B_\zeta^{(-N)}(x,D)\tilde{\Delta}_\zeta \\ &= \tilde{M}_\zeta A_\zeta^{(0)}(x,D)\psi_{\zeta,3}^N(D) - \sum_{k=0}^{N} B_\zeta^{(-k)}(x,D)\psi_{\zeta,3}^{N-k}(D)\tilde{\Delta}_\zeta. \end{aligned}$$

Hence putting

$$A_{\zeta,1}(x,\xi) = A_\zeta^{(0)}(x,\xi)\psi_{\zeta,3}^N(\xi) \text{ and } B_{\zeta,1}(x,\xi) = \sum_{k=0}^{N} B_\zeta^{(-k)}(x,\xi)\psi_{\zeta,3}^{N-k}(\xi), \tag{2.19}$$

we get

$$\tilde{M}_\zeta A_{\zeta,1}(x,D)\chi_{\zeta,1}(D) - B_{\zeta,1}(x,D)\chi_{\zeta,1}(D)\tilde{\Delta}_\zeta = I_\zeta^{(-N)}\chi_{\zeta,1}(D) \in L^{-N}(Z). \tag{2.20}$$

By (2.18,20), we obtain (2.12) with $A_\zeta = A_\zeta(x,D)$ and $B_\zeta = B_\zeta(x,D)$ given by

$$A_\zeta(x,\xi) = \sum_{j=1}^{2} A_{\zeta,j}(x,\xi)\chi_{\zeta,j}(\xi) = A_\zeta^{(0)}(x,\xi) + \sum_{k=1}^{N-1} A_\zeta^{(-k)}(x,\xi)\chi_{\zeta,2}(\xi), \tag{2.21}$$

$$\begin{aligned} B_\zeta(x,\xi) &= \sum_{j=1}^{2} B_{\zeta,j}(x,\xi)\chi_{\zeta,j}(\xi) \\ &= A_\zeta^{(0)}(x,\xi) + \sum_{k=1}^{N} B_\zeta^{(-k)}(x,\xi)\chi_{\zeta,1}(\xi) + \sum_{k=1}^{N-1} A_\zeta^{(-k)}(x,\xi)\chi_{\zeta,2}(\xi), \end{aligned} \tag{2.22}$$

which are elliptic by the expression of $A_\zeta^{(0)}(x,\xi)$. There exist properly supported A'_ζ, B'_ζ such that $A'_\zeta = A_\zeta$, $B'_\zeta = B_\zeta \mod L^{-\infty}(Z)$, so we have proved Lemma 2.2. $\square$

Step 2. Construction of v_ζ.

Fix a $C_0^\infty(\mathbf{R}^3)$-function $\phi_1(x)$ such as $\phi_1 = 1$ on a neighborhood of $\bar{\Omega}$ and choose $\psi \in C_0^\infty(\mathbf{R}^3)$ such as $\psi = 1$ on a neighborhood of $\bar{\Omega}$ and $\phi_1 B_\zeta \tilde{\Delta}_\zeta \psi I_4 = 0$. We take a solution v_ζ to (2.8) of the form

$$v_\zeta = (\alpha\cdot(D+\zeta) + \alpha\cdot a - q^I)\Lambda_\zeta^{-1}A_\zeta(\psi I_4 + w_\zeta), \tag{2.23}$$

here w_ζ satisfies $\phi_1(B_\zeta\tilde{\Delta}_\zeta + R_\zeta^{(-N)})(\psi I_4 + w_\zeta) = 0$, i.e.

$$\phi_1(B_\zeta\tilde{\Delta}_\zeta + R_\zeta^{(-N)})w_\zeta = -\phi_1 R_\zeta^{(-N)}\psi I_4. \tag{2.24}$$

Let us solve (2.24). Put $C_\zeta := B_\zeta\Lambda_\zeta^{-1}$. There exist C_0^∞-functions $\phi_2(x)$, $\phi_3(x)$ such that $\phi_1 C_\zeta \phi_2 = \phi_1 C_\zeta$ and $\phi_1 R_\zeta^{(-N)}\phi_3 = \phi_1 R_\zeta^{(-N)}$, since C_ζ and $R_\zeta^{(-N)}$ are properly supported. Moreover, for $|\zeta|$ large enough, there exists a linear map $\tilde{C}_\zeta^{-1}$ from H^s to H_{loc}^{s-1}, $s \in \mathbf{R}$ such that

$$\phi_1 C_\zeta \tilde{C}_\zeta^{-1} = \phi_1 \text{ and } \|\phi_2\tilde{C}_\zeta^{-1}\|_{s,s-1} \le C_s|\zeta|.$$

Here $\|\cdot\|_{s,s-1}$ is the operator norm from H^s to H^{s-1}. So we solve

$$(\Delta_\zeta + \phi_2\tilde{C}_\zeta^{-1}R_\zeta^{(-N)}\phi_3)w_\zeta = \phi_2\tilde{C}_\zeta^{-1}(-\phi_1 R_\zeta^{(-N)}\psi I_4).$$

We define a linear map Δ_ζ^{-1} from $H_{\delta+1}^m$ to H_δ^m, for any integer $m \ge 0$ and $-1 < \delta < 0$, by

$$\Delta_\zeta^{-1}g = \mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{\xi^2 + 2\zeta\cdot\xi}\right).$$

Then $u = \Delta_\zeta^{-1} g \in H_\delta^m$ is a unique solution of $\Delta_\zeta u = g \in H_{\delta+1}^m$ and we have

$$\|\Delta_\zeta^{-1}\|_{B(H_{\delta+1}^m, H_\delta^m)} \le C_{\delta,m}|\zeta|^{-1},$$

(see Proposition 2.1 and Corollary 2.2 in [SU]). Here $H_\delta^m = H_\delta^m(\mathbf{R}^3)$ is the weighted Sobolev space with norm $\|f\|_{H_\delta^m} = \sum_{|\alpha|\le m} \| < x >^\delta D^\alpha f\|_{L^2(\mathbf{R}^3)}$. Hence (2.24) has a solution of the form

$$\begin{aligned} w_\zeta &= (I + R')^{-1}\Delta_\zeta^{-1}\phi_2\tilde{C}_\zeta^{-1}(-\phi_1 R_\zeta^{(-N)}\psi I_4) \\ \text{with} \quad R' &= \Delta_\zeta^{-1}\phi_2\tilde{C}_\zeta^{-1}R_\zeta^{(-N)}\phi_3, \end{aligned}$$

if $|\zeta|$ large enough and $N \ge 2$, since

$$\begin{aligned} &\|R'\|_{B(H_\delta^m, H_\delta^m)} \\ &\le \|\Delta_\zeta^{-1}\|_{B(H_{\delta+1}^m, H_\delta^m)}\|\phi_2\tilde{C}_\zeta^{-1}\|_{B(H^{m+1}, H_{\delta+1}^m)}\|R_\zeta^{(-N)}\phi_3\|_{B(H_\delta^m, H^{m+1})} \\ &\le C|\zeta|^{-1}\cdot C|\zeta|\cdot C|\zeta|^{-N+1} = C'|\zeta|^{-N+1}. \end{aligned}$$

And similarly we have

$$\|w_\zeta\|_{H_\delta^m} \le C_{\delta,m}|\zeta|^{-N+1}. \tag{2.25}$$

Step 3. Asymptotics of v_ζ.

Lemma 2.3. *Let $A_\zeta \in L^m(Z)$ and $\tilde{\sigma}(A_\zeta) \equiv a_\zeta^{(m)}(x,\xi) + a_\zeta^{(m-1)}(x,\xi)$ mod $S^{m-2}(Z)$ with $a_\zeta^{(m)} \in HS^m(Z)$ and $a_\zeta^{(m-1)} \in HS^{m-1}(Z)$. Let $\phi_1(x)$, $\phi_2(x) \in C_0^\infty(\mathbf{R}^3)$. Then we have for $s, l \in \mathbf{R}$, $m-1 \le l$,*

$$\|\phi_1(A_\zeta - a_\zeta^{(m)}(x,0))\phi_2 f\|_s \le \begin{cases} C_{s,l}|\zeta|^{m-1}\|f\|_{s+l+1}, & (l \le 0) \\ C_{s,l}|\zeta|^{m-1-l}\|f\|_{s+l+1}, & (l \ge 0) \end{cases} \tag{2.26}$$

and for $s, l \in \mathbf{R}$, $m-2 \le l$,

$$\begin{aligned} &\|\phi_1[A_\zeta - a_\zeta^{(m)}(x,0) - a_\zeta^{(m-1)}(x,0) - (\partial_\xi a_\zeta^{(m)})(x,0)\cdot D_x]\phi_2 f\|_s \\ &\le \begin{cases} C_{s,l}|\zeta|^{m-2}\|f\|_{s+l+2}, & (l \le 0) \\ C_{s,l}|\zeta|^{m-2-l}\|f\|_{s+l+2}. & (l \ge 0) \end{cases} \end{aligned} \tag{2.27}$$

Here $a_\zeta^{(m)}(x,0)$, $a_\zeta^{(m-1)}(x,0)$ and $(\partial_\xi a_\zeta^{(m)})(x,0)$ are multiplication operators and $\|\cdot\|_s = \|\cdot\|_{H^s}$.

Proof. Since

$$\begin{aligned} \tilde{\sigma}(A_\zeta)(x,\xi) - a_\zeta^{(m)}(x,0) &\equiv a_\zeta^{(m)}(x,\xi) - a_\zeta^{(m)}(x,0) \\ &= \int_0^1 (\partial_\xi a_\zeta^{(m)})(x,\theta\xi)\,d\theta\cdot\xi \quad \text{mod } S^{m-1}(Z), \\ \text{and} \quad b_\zeta^{(m-1)}(x,\xi) &:= \int_0^1 (\partial_\xi a_\zeta^{(m)})(x,\theta\xi)\,d\theta \in HS^{m-1}(Z), \end{aligned}$$

it follows that, with some $r_\zeta^{(m-1)} \in L^{m-1}(Z)$,

$$\phi_1(A_\zeta - a_\zeta^{(m)}(x,0))\phi_2 f = \phi_1 b_\zeta^{(m-1)}(x,D)\cdot D(\phi_2 f) + \phi_1 r_\zeta^{(m-1)}\phi_2 f.$$

And we apply Theorem 9.1 in [Sh] to get (2.26):

$$\begin{aligned}\|\phi_1(A_\zeta - a_\zeta^{(m)}(x,0))\phi_2 f\|_s &\le \|\phi_1 b_\zeta^{(m-1)}(x,D)\cdot D(\phi_2 f)\|_s + \|\phi_1 r_\zeta^{(m-1)}\phi_2 f\|_s \\ &\le \begin{cases} C_{s,l}|\zeta|^{m-1}\|f\|_{s+l+1}, & (l\le 0) \\ C_{s,l}|\zeta|^{m-1-l}\|f\|_{s+l+1}. & (l \ge 0)\end{cases}\end{aligned}$$

Similarly as above, since we can write

$$\begin{aligned}&\tilde{\sigma}(A_\zeta)(x,\xi) - a_\zeta^{(m)}(x,0) - a_\zeta^{(m-1)}(x,0) - (\partial_\xi a_\zeta^{(m)})(x,0)\cdot\xi \\ &\equiv b_\zeta^{(m-2)}(x,\xi)\cdot\xi + \sum_{j,k=1}^{3} b_{j,k,\zeta}^{(m-2)}(x,\xi)\xi_j\xi_k \quad \mathrm{mod}\ S^{m-2}(Z)\end{aligned}$$

$$\begin{aligned}\text{with}\quad b_\zeta^{(m-2)}(x,\xi) &= \int_0^1 (\partial_\xi a_\zeta^{(m-1)})(x,\theta\xi)\,d\theta \in HS^{m-2}(Z), \\ b_{j,k,\zeta}^{(m-2)}(x,\xi) &= \int_0^1 (1-\theta)(\partial_{\xi_j}\partial_{\xi_k} a_\zeta^{(m)})(x,\theta\xi)d\theta \in HS^{m-2}(Z),\end{aligned}$$

so it suffices to apply Theorem 9.1 in [Sh] to get (2.27). □

We define a function φ_ζ by

$$\varphi_\zeta(x) := -\mathcal{F}^{-1}\left(\frac{\zeta\cdot\hat{a}(\xi)}{\zeta\cdot\xi}\right)(x), \tag{2.28}$$

then $\{\varphi_\zeta\}_{\zeta\in Z}$ is bounded in $\mathcal{B}^\infty(\mathbf{R}^3)$ and φ_ζ satisfies

$$\zeta\cdot(a(x) + D\varphi_\zeta(x)) = 0, \tag{2.29}$$

(cf. [Su]).

Lemma 2.4. *The solution v_ζ in (2.23) has the following asymptotics: for any integer $m \ge 0$,*

$$v_\zeta = \frac{\alpha\cdot\zeta}{|\zeta|}e^{\varphi_\zeta(x)} + (\alpha\cdot(a + D\varphi_\zeta) - q^I)\frac{e^{\varphi_\zeta(x)}}{|\zeta|} + \frac{\alpha\cdot\zeta}{|\zeta|}X_\zeta(x) + O(|\zeta|^{-2}), \quad (|\zeta|\to\infty), \tag{2.30}$$

in $H^m(\Omega)$, with some 4×4-matrix $X_\zeta(x)$ satisfying $\|X_\zeta\|_{H^m(\Omega)} \le C_m|\zeta|^{-1}$.

Note that the first term in (2.30) is $O(1)$, and the second and the third are $O(|\zeta|^{-1})$.

Proof. Let $N = 2$. By (2.21) we have

$$\tilde{\sigma}((\alpha\cdot(D+\zeta) + \alpha\cdot a - q^I)\Lambda_\zeta^{-1}A_\zeta) \equiv e_\zeta^{(0)}(x,\xi) + e_\zeta^{(-1)}(x,\xi) \quad \mathrm{mod}\ S^{-2}(Z),$$

where

$$\begin{aligned}
e_\zeta^{(0)}(x,\xi) &= \alpha\cdot(\xi+\zeta)\lambda_\zeta^{-1}(\xi)A_\zeta^{(0)}(x,\xi) \quad \in HS^0(Z),\\
e_\zeta^{(-1)}(x,\xi) &= \alpha\cdot(\xi+\zeta)\lambda_\zeta^{-1}(\xi)A_\zeta^{(-1)}(x,\xi)\chi_{\zeta,2}(\xi)\\
&\quad + (\alpha\cdot b_\zeta - q^I)\lambda_\zeta^{-1}(\xi)A_\zeta^{(0)}(x,\xi)\\
&\quad + \partial_\xi(\alpha\cdot(\xi+\zeta)\lambda_\zeta^{-1}(\xi))\cdot D_x A_\zeta^{(0)}(x,\xi) \quad \in HS^{-1}(Z).
\end{aligned}$$

Hence applying (2.27) in Lemma 2.3 as $l=m=0$, we have

$$\begin{aligned}
v_\zeta &= [e_\zeta^{(0)}(x,0)+e_\zeta^{(-1)}(x,0)+(\partial_\xi e_\zeta^{(0)})(x,0)\cdot D_x](\psi I_4+w_\zeta)+O(|\zeta|^{-2})\\
&= e_\zeta^{(0)}(x,0)\psi I_4 + [e_\zeta^{(-1)}(x,0)+(\partial_\xi e_\zeta^{(0)})(x,0)\cdot D_x]\psi I_4\\
&\qquad + e_\zeta^{(0)}(x,0)w_\zeta + O(|\zeta|^{-2})\\
&= e_\zeta^{(0)}(x,0)\psi I_4 + e_\zeta^{(-1)}(x,0)\psi I_4 + e_\zeta^{(0)}(x,0)w_\zeta + O(|\zeta|^{-2}).
\end{aligned}$$

Moreover (2.15,16) yield

$$\begin{aligned}
e_\zeta^{(0)}(x,0) &= \frac{\alpha\cdot\zeta}{|\zeta|}A_\zeta^{(0)}(x,0) = \frac{\alpha\cdot\zeta}{|\zeta|}e^{\varphi_\zeta(x)},\\
e_\zeta^{(-1)}(x,0) &= \frac{\alpha\cdot\zeta}{|\zeta|}A_\zeta^{(-1)}(x,0) + (\alpha\cdot a - q^I)\frac{e^{\varphi_\zeta(x)}}{|\zeta|} + \frac{1}{|\zeta|}(\alpha\cdot D_x A_\zeta^{(0)})(x,0)\\
&= (\alpha\cdot(a+D\varphi_\zeta)-q^I)\frac{e^{\varphi_\zeta(x)}}{|\zeta|} + \frac{\alpha\cdot\zeta}{|\zeta|}A_\zeta^{(-1)}(x,0).
\end{aligned}$$

Hence putting $X_\zeta(x) = e^{\varphi_\zeta(x)}w_\zeta(x) + A_\zeta^{(-1)}(x,0)$, by (2.25) we get Lemma 2.4. □

Step 4. Proof of $\mathrm{rot}a_1 = \mathrm{rot}a_2$ and $q_1 = q_2$.

The rest of the proof of Theorem 1 is basically the same as in [T1], but we repeat it to make the proof self-contained.

Fix $k \neq 0, \eta, \gamma \in \mathbf{R}^3$ such as $k\cdot\eta = k\cdot\gamma = \eta\cdot\gamma = 0$, $|\eta| = |\gamma| = 1$, and define $\{\zeta_j(\lambda)\}_{\lambda>1} \subset Z$, $j=1,2$, by

$$\begin{cases}
\zeta_1 = \zeta_1(\lambda) = \lambda(\omega_1(\lambda)+i\gamma), \quad \omega_1(\lambda) = (1-\dfrac{k^2}{4\lambda^2})^{1/2}\eta - \dfrac{k}{2\lambda},\\
\zeta_2 = \zeta_2(\lambda) = \lambda(\omega_2(\lambda)-i\gamma), \quad \omega_2(\lambda) = (1-\dfrac{k^2}{4\lambda^2})^{1/2}\eta + \dfrac{k}{2\lambda}.
\end{cases}$$

Note that $\zeta_1^2 = \zeta_2^2 = 0$, $\overline{\zeta_2} - \zeta_1 = k$ and $\frac{\zeta_1}{\lambda}, \frac{\overline{\zeta_2}}{\lambda} \to \zeta_0 \equiv \eta + i\gamma\,(\lambda\to\infty)$. We substitute the solution $u_{\zeta_j} = e^{i\zeta_j\cdot x}v_{\zeta_j}$ of $L_{a_j,q_j}u_{\zeta_j} = 0$, $j=1,2$, for $u^{(j)}$ of (2.6) to get

$$K(\lambda) := \int_\Omega e^{-ik\cdot x}v_{\zeta_2}^*(V_1-V_2)v_{\zeta_1}\,dx = 0,$$

here A^* denotes the adjoint matrix of A.

First we show $\mathrm{rot}a_1 = \mathrm{rot}a_2$. By Lemma 2.4, we have

$$K(\lambda) = \int_\Omega e^{-ik\cdot x+\varphi_1+\overline{\varphi_2}} \frac{(\alpha\cdot\zeta_2)^*}{|\zeta_2|}(V_1 - V_2)\frac{\alpha\cdot\zeta_1}{|\zeta_1|}\,dx + O(\lambda^{-1}), \quad (\lambda\to\infty)$$

here

$$\varphi_j = -\mathcal{F}^{-1}\Big(\frac{\zeta_j\cdot\hat{a}_j(\xi)}{\zeta_j\cdot\xi}\Big), \quad j=1,2.$$

Using $(\alpha\cdot\zeta_2)^* = \alpha\cdot\overline{\zeta_2}$ and $\zeta_1/|\zeta_1|$, $\overline{\zeta_2}/|\zeta_2| \to \zeta_0/\sqrt{2}$ and

$$\varphi_1 + \overline{\varphi_2} \to \psi := -\mathcal{F}^{-1}\Big(\frac{\zeta_0\cdot((\hat{a}_1-\hat{a}_2)(\xi))}{\zeta_0\cdot\xi}\Big), \quad (\lambda\to\infty)$$

we get

$$\begin{aligned} K(\lambda) \to& \frac{1}{2}\int_\Omega e^{-ik\cdot x+\psi}\alpha\cdot\zeta_0(V_1-V_2)\alpha\cdot\zeta_0\,dx \\ =&\alpha\cdot\zeta_0\int_\Omega e^{-ik\cdot x+\psi}\zeta_0\cdot(a_1-a_2)dx. \quad (\lambda\to\infty)\end{aligned}$$

Since $\alpha\cdot\zeta_0\neq 0$, it follows that

$$\int_\Omega e^{-ik\cdot x+\psi}\zeta_0\cdot(a_1-a_2)dx = 0.$$

This yields $\mathrm{rot}a_1 = \mathrm{rot}a_2$ by arguments in [Su,§4].

Next we show $q_1 = q_2$. Since $\mathrm{rot}a_1 = \mathrm{rot}a_2$ and Γ is connected, there exists $p \in C^\infty(\mathbf{R}^3)$ such that $a_1 - a_2 = \nabla p$ and $p|_\Gamma = 0$. Hence by the gauge invariance, $\Lambda_{a_1,q_1} = \Lambda_{a_2,q_2}$ implies $\Lambda_{a_1,q_1} = \Lambda_{a_1,q_2}$. So we may assume $a_1 = a_2 =: a$ to prove $q_1 = q_2$.

Lemma 2.5.

$$P_\pm\lambda K(\lambda)P_\pm \to \frac{\alpha\cdot k}{2}\int_\Omega e^{-ik\cdot x}P_\mp(q_1-q_2)P_\mp dx\,\alpha\cdot\zeta_0, \quad (\lambda\to\infty).$$

Once this is proved, it is easy to see $q_1 = q_2$.

Proof. Put $q := q_1 - q_2$ and $b_{\zeta_j} := a + D\varphi_{\zeta_j}$ $j = 1,2$. By Lemma 2.4, we have

$$\begin{aligned}\lambda K(\lambda) = \lambda&\int_\Omega e^{-ik\cdot x}\Big(\frac{\alpha\cdot\zeta_2}{|\zeta_2|}e^{\varphi_2} + (\alpha\cdot b_{\zeta_2} - q_2^I)\frac{e^{\varphi_2}}{|\zeta_2|} + \frac{\alpha\cdot\zeta_2}{|\zeta_2|}X_{\zeta_2}\Big)^* \\ &\times q\Big(\frac{\alpha\cdot\zeta_1}{|\zeta_1|}e^{\varphi_1} + (\alpha\cdot b_{\zeta_1} - q_1^I)\frac{e^{\varphi_1}}{|\zeta_1|} + \frac{\alpha\cdot\zeta_1}{|\zeta_1|}X_{\zeta_1}\Big)\,dx + O(\lambda^{-1}) \\ =\lambda&\int_\Omega e^{-ik\cdot x}\Big(\frac{\alpha\cdot\zeta_2}{|\zeta_2|}e^{\varphi_2}\Big)^* q\frac{\alpha\cdot\zeta_1}{|\zeta_1|}e^{\varphi_1}\,dx \\ +\lambda&\int_\Omega e^{-ik\cdot x}\Big(\frac{\alpha\cdot\zeta_2}{|\zeta_2|}e^{\varphi_2}\Big)^* q\Big((\alpha\cdot b_{\zeta_1} - q_1^I)\frac{e^{\varphi_1}}{|\zeta_1|} + \frac{\alpha\cdot\zeta_1}{|\zeta_1|}X_{\zeta_1}\Big)\,dx \\ +\lambda&\int_\Omega e^{-ik\cdot x}\Big((\alpha\cdot b_{\zeta_2} - q_2^I)\frac{e^{\varphi_2}}{|\zeta_2|} + \frac{\alpha\cdot\zeta_2}{|\zeta_2|}X_{\zeta_2}\Big)^* q\frac{\alpha\cdot\zeta_1}{|\zeta_1|}e^{\varphi_1}\,dx + O(\lambda^{-1}) \\ =\frac{1}{2}&\int_\Omega e^{-ik\cdot x}[\alpha\cdot kq\alpha\cdot\zeta_0 + \alpha\cdot\zeta_0 q(\alpha\cdot b_{\zeta_0} - q_1^I) + (\alpha\cdot b_{\zeta_0} - q_2^I)q\alpha\cdot\zeta_0]\,dx \\ &+ O(\lambda^{-1}),\end{aligned}$$

where we have used $\overline{\zeta_2} = \zeta_1 + k$ and $\varphi_1 + \overline{\varphi_2} \to 0$ and $b_{\zeta_1}, \overline{b_{\zeta_2}} \to b_{\zeta_0}$ $(\lambda \to \infty)$ in the last step. Together with $\alpha \cdot \zeta_0 q \alpha \cdot b_{\zeta_0} + \alpha \cdot b_{\zeta_0} q \alpha \cdot \zeta_0 = 0$ (by (2.5,29)), we get

$$\lambda K(\lambda) \to \frac{1}{2} \int_\Omega e^{-ik\cdot x}(\alpha \cdot k\, q\alpha \cdot \zeta_0 - \alpha \cdot \zeta_0 q q_1^I - q_2^I q\, \alpha \cdot \zeta_0) dx.$$

This and (2.4) yield

$$P_\pm \lambda K(\lambda) P_\pm \to \frac{1}{2} \int_\Omega e^{-ik\cdot x} \alpha \cdot k\, P_\mp q P_\mp \alpha \cdot \zeta_0 dx.$$

□

3. Proof of Theorem 2

Under a condition such as scalar potential q does not vanish at the boundary, we can prove the uniqueness at the boundary (Theorem 2 in [NT]), by expressing the D-D map $\Lambda_{\vec{a},q}$ by the asymptotic expansion of the pseudodifferential operator. Here the constraint on scalar potential can be removed by applying the method of [A], in which uniqueness and stability of inverse problems for conductivity at the boundary was obtained. We will construct singular solutions of Dirac equation, and approach the singularity to the boundary to get informations of potentials. However we need a different choice of the leading term of singular solution from [A]: in which, harmonic spherical functions S_m are chosen through the Gegenbauer polynomials, while ours come from associated Legendre functions Y_m^m. On the other hand, uniqueness of scalar potential q at the boundary can be seen by the same choice of S_m and arguments as in [A], moreover uniqueness of q on Ω is known in Theorem 1, so we will not discuss about it here.

Let $B_R(x_0) = \{x \in \mathbf{R}^3;\ |x - x_0| < R\}$ be a ball of radius R and center x_0. In this section, write $B_R = B_R(0)$ and assume $\vec{a},\ q \in C^\infty(\overline{B_R})$.

Proposition 3.1. *(singular solutions)*
For any spherical harmonic S_m of degree $m = 0, 1, 2, \cdots$, there exists 4×4 matrix valued $u(x) \in L^\infty_{loc}(B_R \setminus \{0\})$ such that $L_{\vec{a},q} u = 0$ in $B_R \setminus \{0\}$, and u is of the form

$$u(x) = \alpha \cdot D_x \left(|x|^{-1-m} S_m(\frac{x}{|x|}) \right) + v(x),$$

and $v(x)$ satisfies $|v(x)| \le C|x|^{-2-m+\varepsilon}$, for any $0 < \varepsilon < 1$. Here, C depends only on $S_m, \vec{a}, q, R, \varepsilon$.

Proof is omitted.

We define a phase function $p_j(x) \in C^\infty(\overline{\Omega})$, $j = 1, 2$, near $\partial\Omega$, by

$$p_j(x) = \int_0^{l(x)} N(\pi(x)) \cdot \vec{a}_j(\pi(x) - sN(\pi(x)))ds, \quad j = 1, 2,$$

where $N(x)$ is outer unit normal at $x \in \partial\Omega$, and the projection $\pi(x) \in \partial\Omega$ and the distance $l(x) \ge 0$ are uniquely taken such as $x - \pi(x) = -l(x)N(\pi(x))$. Note $p_j|_{\partial\Omega} = 0$.

Set $\vec{b}(x) = \vec{a}_1(x) - \vec{a}_2(x) - \nabla(p_1 - p_2)(x)$. We will show that $\partial_x^\alpha \vec{b}|_{\partial\Omega} = 0$ for any multi-index α, by induction on $|\alpha| = k \ge 0$. If $\Lambda_{\vec{a}_1,q_1} = \Lambda_{\vec{a}_2,q_2}$, then $\Lambda_{\vec{a}_1 - \nabla p_1, q_1} = \Lambda_{\vec{a}_2 - \nabla p_2, q_2}$ by the gauge invariance. So by Lemma 2.1, we have the key identity:

$$0 = \int_\Omega u_2^*(x)(\alpha \cdot \vec{b}(x) + q_1(x) - q_2(x))u_1(x)dx, \tag{3.1}$$

with a solution u_j to $L_{\vec{a}_j-\nabla p_j,q_j}u_j=0$, $j=1,2$.

Fix $x_0\in\partial\Omega$. By a translation and a rotation, we introduce new coordinates: $x'=R(x-x_0)$, where a rotation matrix $R=(R_{kl})$ is chosen such that $\partial\Omega$ is tangent to $(x_1'x_2')$-plane at $x'=0$ and $\{0<x_3'<\delta_0,\ x_1'=x_2'=0\}\subset\Omega$ for small $\delta_0>0$. By the change of variables, (3.1) is rewritten as

$$0=\int_{\Omega'}u_2'^{*}(x')(\alpha'\cdot\vec{b}'(x')+q_1'(x')-q_2'(x'))u_1'(x')dx',\tag{3.1'}$$

where $u_j'(x')$ satisfies

$$[\alpha'\cdot(D_{x'}+\vec{a}_j'(x')-\nabla_{x'}p_j'(x'))+q_j'(x')]u_j'(x')=0,$$

where

$$\begin{aligned}
&u_j'(x')=u_j(R^{-1}x'+x_0),\\
&\vec{a}_j'=(a_{j1}',a_{j2}',a_{j3}'),\quad a_{jk}'(x')=\sum_{l=1}^{3}R_{kl}a_{jl}(R^{-1}x'+x_0),\\
&\vec{b}'=(b_1',b_2',b_3'),\quad b_k'(x')=\sum_{l=1}^{3}R_{kl}b_l(R^{-1}x'+x_0),\\
&\alpha_k'=\sum_{l=1}^{3}R_{kl}\alpha_l,\qquad k=1,2,3,\\
&p_j'(x')=p_j(R^{-1}x'+x_0),\quad q_j'(x')=q_j(R^{-1}x'+x_0).
\end{aligned}$$

Note that $\sigma_k'=\sum_{l=1}^{3}R_{kl}\sigma_l$, $k=1,2,3$, also satisfy the relations:

$$\begin{aligned}
&\sigma_j'\sigma_k'+\sigma_k'\sigma_j'=2\delta_{jk}I_2,\quad j,k=1,2,3,\\
&\sigma_1'\sigma_2'=i\sigma_3',\quad \sigma_2'\sigma_3'=i\sigma_1',\quad \sigma_3'\sigma_1'=i\sigma_2'.
\end{aligned}$$

First we will show $\vec{b}'(0)=0$, which means $\vec{b}(x_0)=0$, and then, since $x_0\in\partial\Omega$ is arbitrary, $\vec{b}|_{\partial\Omega}=0$ follows. It is clear that $b_3'(0)=0$, by the definition. In the following arguments we omit the symbol "$'$" of $x',u_j',\alpha',\vec{a}_j',\vec{b}',p_j',q_j'$.

Fix $R>2\mathrm{diam}\Omega$ and let $\delta>0$ be small such as $B_R(x_\delta)\supset\Omega$, here $x_\delta:=(0,0,-\delta)$. We can extend $\vec{a}_j,p_j,q_j\in C^\infty(\overline{\Omega})$, $j=1,2$, such as $\vec{a}_j,p_j,q_j\in C^\infty(\overline{B_R(x_\delta)})$. By Proposition 2.3, we can take u_j as

$$u_j(x)=\alpha\cdot D_x\left(|x-x_\delta|^{-1-m}S_m(\frac{x-x_\delta}{|x-x_\delta|})\right)+v_j(x)\in L^\infty_{loc}(B_R(x_\delta)\setminus x_\delta),$$

with some v_j satisfying $|v_j(x)|\le C|x-x_\delta|^{-2-m+\varepsilon}$. Take $S_m(x/|x|)=(x_1+ix_2)^m/|x|^m=z^m/|x|^m$, $(z:=x_1+ix_2)$, and put $\vec{d}(x)=(d_1(x),d_2(x),d_3(x))$, with $d_k(x)=\partial_{x_k}(|x|^{-1-m}S_m(x/|x|))$, $k=1,2,3$.

From (3.1), we obtain

$$\begin{aligned}&\int_\Omega (\alpha\cdot\vec{d}(x-x_\delta))^*\alpha\cdot\vec{b}(x)\alpha\cdot\vec{d}(x-x_\delta)dx\\&=\int_\Omega v_2^*(x)V(x)\alpha\cdot\vec{d}(x-x_\delta)+(\alpha\cdot\vec{d}(x-x_\delta))^*V(x)v_1(x)+v_2^*(x)V(x)v_1(x)dx,\end{aligned}$$

here $V(x)=\alpha\cdot\vec{b}(x)+q_1(x)-q_2(x)$, hence it is easy to see

$$\left|\int_\Omega (\sigma\cdot\vec{d}(x-x_\delta))^*\sigma\cdot\vec{b}(x)\sigma\cdot\vec{d}(x-x_\delta)dx\right|\le C\int_\Omega |x-x_\delta|^{-4-2m+\varepsilon}dx\le C\delta^{-1-2m+\varepsilon}.$$

Here $|A|=\sum_{i,j}|a_{ij}|$ for a matrix $A=(a_{ij})$. Since $|\vec{b}(x)-\vec{b}(0)|\le\|\nabla\vec{b}\|_{L^\infty(\Omega)}|x|$, it follows that

$$\begin{aligned}&\left|\int_\Omega (\sigma\cdot\vec{d}(x-x_\delta))^*\sigma\cdot\vec{b}(0)\sigma\cdot\vec{d}(x-x_\delta)dx\right|\\&\le C\|\nabla\vec{b}\|_{L^\infty(\Omega)}\int_\Omega |x||x-x_\delta|^{-4-2m}dx+C\delta^{-1-2m+\varepsilon}\\&\le C\delta^{-1-2m+\varepsilon}.\end{aligned}$$

Changing the domain of integration, we have

$$\begin{aligned}&\left|\int_{\{x_3\ge 0\}\cap B_R(x_\delta)} (\sigma\cdot\vec{d}(x-x_\delta))^*\sigma\cdot\vec{b}(0)\sigma\cdot\vec{d}(x-x_\delta)dx\right|\\&\le C|\sigma\cdot\vec{b}(0)|\int_{\Omega\triangle(\{x_3\ge 0\}\cap B_R(x_\delta))}|x-x_\delta|^{-4-2m}dx+C\delta^{-1-2m+\varepsilon}\\&\le C\delta^{-1-2m+\varepsilon},\end{aligned}$$

where we have used Lemma 3.2 below in the last step (we should take $m\ge 1$), and put $A\triangle B:=(A\setminus B)\cup(B\setminus A)$.

By direct caluculation, using the relations of Pauli matrices and $b_3(0)=0$, we have

$$\begin{aligned}(\sigma\cdot\vec{d})^*\sigma\cdot\vec{b}\sigma\cdot\vec{d}=&\ \sigma_1[b_1(|d_1|^2-|d_2|^2-|d_3|^2)+2b_2\mathrm{Re}\,(d_1\bar{d}_2)]\\&+\sigma_2[b_2(-|d_1|^2+|d_2|^2-|d_3|^2)+2b_1\mathrm{Re}\,(d_1\bar{d}_2)]\\&+\sigma_3[2b_1\mathrm{Re}\,(d_1\bar{d}_3)+2b_2\mathrm{Re}\,(d_2\bar{d}_3)]+[-2b_1\mathrm{Im}\,(d_2\bar{d}_3)+2b_2\mathrm{Im}\,(d_1\bar{d}_3)]\end{aligned}$$

and

$$\begin{aligned}&|d_1(x)|^2-|d_2(x)|^2\\&\quad=(x_1^2-x_2^2)[(1+2m)^2|x|^{-6-4m}|z|^{2m}+2m(-1-2m)|x|^{-4-4m}|z|^{2(m-1)}],\\&|d_3(x)|^2=(1+2m)^2x_3^2|x|^{-6-4m}|z|^{2m},\\&\mathrm{Re}\,(d_1\bar{d}_2)(x)=x_1x_2[(1+2m)^2|x|^{-6-4m}|z|^{2m}+2m(-1-2m)|x|^{-4-4m}|z|^{2(m-1)}],\\&(d_1\bar{d}_3)(x)=x_1x_3(1+2m)^2|x|^{-6-4m}|z|^{2m}+x_3\bar{z}m(-1-2m)|x|^{-4-4m}|z|^{2(m-1)},\\&(d_2\bar{d}_3)(x)=x_2x_3(1+2m)^2|x|^{-6-4m}|z|^{2m}+ix_3\bar{z}m(-1-2m)|x|^{-4-4m}|z|^{2(m-1)}.\end{aligned}$$

Hence

$$\int_{\{x_3\geq 0\}\cap B_R(x_\delta)} (\sigma\cdot\vec{d}(x-x_\delta))^*\sigma\cdot\vec{b}(0)\sigma\cdot\vec{d}(x-x_\delta)dx = -\sigma\cdot\vec{b}(0)\int_{\{x_3\geq 0\}\cap B_R(x_\delta)} |d_3(x-x_\delta)|^2.$$

Moreover, since

$$\int_{\{x_3\geq 0\}\cap B_R(x_\delta)} |d_3(x-x_\delta)|^2 \geq C\delta^{-1-2m},$$

it follows that

$$C\delta^{-1-2m}|\sigma\cdot\vec{b}(0)| \leq C\delta^{-1-2m+\varepsilon},$$

hence $|\sigma\cdot\vec{b}(0)| = 0$, and so $b_1(0) = b_2(0) = 0$.

Next suppose that the induction hypothesis:

$$\partial_x^\alpha \vec{b}(x) = 0, \text{ on } \partial\Omega, \quad 0 \leq |\alpha| \leq k-1. \tag{3.2}$$

Then it is easy to see

$$\partial_{x_l}\partial_x^\alpha \vec{b}(0) = 0, \quad 0 \leq |\alpha| \leq k-1, \quad l = 1, 2. \tag{3.3}$$

We will show $\partial_{x_3}^k\vec{b}(0) = 0$, which yields $\partial_x^\alpha\vec{b}(0) = 0$, $|\alpha| = k$, and hence $\partial_x^\alpha\vec{b}|_{\partial\Omega} = 0$, $|\alpha| = k$, as before. From (3.2) and (3.3), we have

$$|\vec{b}(x) - x_3^k\partial_{x_3}^k\vec{b}(0)/k!| \leq M|x|^{k+1}, \quad x \in \Omega, \tag{3.4}$$

and

$$|\vec{b}(x)| \leq M'|x|^k, \quad x \in \Omega. \tag{3.5}$$

From the key identity (3.1), it follows that, by (3.5)

$$\begin{aligned} &\left|\int_\Omega (\sigma\cdot\vec{d}(x-x_\delta))^*\sigma\cdot\vec{b}(x)\sigma\cdot\vec{d}(x-x_\delta)dx\right| \\ &\leq CM'\int_\Omega |x|^k|x-x_\delta|^{-4-2m+\varepsilon}dx \leq C\delta^{-1-2m+\varepsilon+k}, \end{aligned}$$

and hence, by (3.4)

$$\begin{aligned} &\left|\int_\Omega (\sigma\cdot\vec{d}(x-x_\delta))^* x_3^k\sigma\cdot\partial_{x_3}^k\vec{b}(0)\sigma\cdot\vec{d}(x-x_\delta)dx\right| \\ &\leq CM\int_\Omega |x|^{k+1}|x-x_\delta|^{-4-2m}dx + C\delta^{-1-2m+\varepsilon+k} \\ &\leq C\delta^{-1-2m+\varepsilon+k}. \end{aligned}$$

Changing the domain of integration, we have

$$\begin{aligned} &\left|\int_{\{x_3\geq 0\}\cap B_R(x_\delta)} (\sigma\cdot\vec{d}(x-x_\delta))^* x_3^k\sigma\cdot\partial_{x_3}^k\vec{b}(0)\sigma\cdot\vec{d}(x-x_\delta)dx\right| \\ &\leq C|\sigma\cdot\partial_{x_3}^k\vec{b}(0)|\int_{\Omega\triangle(\{x_3\geq 0\}\cap B_R(x_\delta))} |x_3|^k|x-x_\delta|^{-4-2m}dx + C\delta^{-1-2m+\varepsilon+k} \\ &\leq C\delta^{-1-2m+\varepsilon+k}, \end{aligned}$$

where in the last step we have used $|x_3| \le |x - x_\delta|$ and Lemma 3.2 below (we should take $m > k/2$). The same caluculation as before yields (note $\partial_{x_3}^k b_3(0) = 0$)

$$\begin{aligned}
&\int_{\{x_3 \ge 0\} \cap B_R(x_\delta)} (\sigma \cdot \vec{d}(x - x_\delta))^* x_3^k \sigma \cdot \partial_{x_3}^k \vec{b}(0) \sigma \cdot \vec{d}(x - x_\delta) dx \\
&= -\sigma \cdot \partial_{x_3}^k \vec{b}(0) \int_{\{x_3 \ge 0\} \cap B_R(x_\delta)} |d_3(x - x_\delta)|^2 x_3^k dx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\{x_3 \ge 0\} \cap B_R(x_\delta)} |d_3(x - x_\delta)|^2 x_3^k dx \\
&= (1 + 2m)^2 \int_{\{x_3 \ge 0\} \cap B_R(x_\delta)} |x - x_\delta|^{-6-4m} |x_3 + \delta|^2 |z|^{2m} x_3^k dx \\
&\ge C\delta^{2+2m+k} \int_{\{x_3 \ge \delta\} \cap B_R(x_\delta) \cap \{|z| \ge \delta\}} |x - x_\delta|^{-6-4m}\, dx \\
&\ge C\delta^{-1-2m+k}.
\end{aligned}$$

Consequently we have

$$C\delta^{-1-2m+k} |\sigma \cdot \partial_{x_3}^k \vec{b}(0)| \le C\delta^{-1-2m+\varepsilon+k},$$

hence $\partial_{x_3}^k b_1(0) = \partial_{x_3}^k b_2(0) = 0$. Therefore we have proved the theorem. □

Lemma 3.2. *Let $s > 4$. We have*

$$\int_{\Omega \triangle (\{x_3 \ge 0\} \cap B_R(x_\delta))} |x - x_\delta|^{-s} dx \le C\delta^{-s+4}, \quad \textit{for } \delta \ll 1.$$

Proof. Near the origin, let $\partial\Omega$ be represented by $x_3 = \varphi(x_1, x_2)$ and Ω be represented by $x_3 > \varphi(x_1, x_2)$. Since $\partial\Omega$ is smooth, there exist constants $c_0 > 0$ and $\rho > 0$, such that $|\varphi(x_1, x_2)| \le c_0(x_1^2 + x_2^2)$ for $(x_1^2 + x_2^2) \le \rho$. Therefore it suffices to show

$$\int_{\{|x_3| \le c_0(x_1^2 + x_2^2) \le c_0\rho\}} |x - x_\delta|^{-s} dx \le C\delta^{-s+4}, \quad \text{for } \delta \ll 1. \tag{3.6}$$

The left hand side of (3.6) is bounded by

$$\begin{aligned}
\text{L.H.S. of (3.6)} &\le C \int_0^\rho r\, dr \int_0^{c_0 r^2} (r^2 + (\delta - t)^2)^{-s/2}\, dt \\
&= C\delta^{-s+3} \int_0^{\rho/\delta} r^{-s+2}\, dr \int_{1/r - c_0\delta r}^{1/r} (1 + t^2)^{-s/2}\, dt \\
&= C\delta^{-s+3} \Bigg[\int_0^{\rho\delta^{1/(-s+3)}} r^{-s+2}\, dr \int_{1/r - c_0\delta r}^{1/r} (1 + t^2)^{-s/2}\, dt \\
&\qquad + \int_{\rho\delta^{1/(-s+3)}}^{\rho/\delta} r^{-s+2}\, dr \int_{1/r - c_0\delta r}^{1/r} (1 + t^2)^{-s/2}\, dt \Bigg].
\end{aligned}$$

In the first term of the above, it follows that

$$\int_{1/r-c_0\delta r}^{1/r} (1+t^2)^{-s/2}\,dt \le C\delta r(1+r^{-2})^{-s/2}, \quad \text{for } \delta \ll 1,$$

and in the second term

$$\int_{1/r-c_0\delta r}^{1/r} (1+t^2)^{-s/2}\,dt \le C.$$

Hence

$$\begin{aligned}\text{L.H.S. of (3.6)} &\le C\delta^{-s+3}\left[\int_0^\infty r^{-s+2}\delta r(1+r^{-2})^{-s/2}\,dr + \int_{\rho\delta^{1/(-s+3)}}^\infty r^{-s+2}\,dr\right]\\ &\le C\delta^{-s+4}.\end{aligned}$$

So we have proved the lemma. $\square$

References

[A] G.Alessandrini, Singular solutions of elliptic equations and the determination of conductivity by boundary measurements, J. Differential Equations **84** (1990), pp. 252–272.

[G] M.Goto, Inverse scattering for Dirac operators with electro-magnetic potentials at a fixed energy.(1999).

[I] H.Isozaki, Inverse scattering theory for Dirac operators, Ann. Inst. Henri Poincare, Phys. Theor. **66** (1997), pp. 237–270.

[NSU] G.Nakamura, Z.Sun, G.Uhlmann, Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field, Math. Ann. **303** (1995), pp. 377–388.

[NT] G.Nakamura, T.Tsuchida, Uniqueness for an inverse boundary value problem for Dirac operators, (1999) to appear in Comm. PDE.

[NU] G.Nakamura, G.Uhlmann, Global uniqueness for an inverse boundary problem arising in elasticity, Invent. Math. **118** (1994), pp. 457–474.

[Sh] M.A.Shubin, *Pseudodifferential operators and spectral theory*, Springer Series in Soviet Mathematics, Springer-Verlag, 1987.

[Su] Z.Sun, An inverse boundary value problem for Schrödinger operators with vector potentials, Trans. of AMS, **338** (1993), pp. 953–969.

[SU] J.Sylvester, G.Uhlmann, A global uniqueness theorem for an inverse bounbary value problem, Ann. of Math. (2) **125** (1987), pp.153–169.

[T1] T.Tsuchida, An inverse boundary value problem for Dirac operators with small potentials, Kyushu J. Math. **52** (1998), pp. 361–382.

[T2] T.Tsuchida, Uniqueness at the boundary for an inverse problem for Dirac operators, preprint.